18F05.

Lecture 5.

Almost matematics I. Chapter 4 of Bhatt.

A. Some context, abelian categories.

$\mathcal{A} \underset{\text{f.f.}}{\subseteq} \mathcal{B}$ abelian categories.

We say $\mathcal{A}$ is a Serre subcat. of $\mathcal{B}$ if it is closed under extensions, subobjects, quotient objects.

Exs (i) X a noetherian scheme, $Z \subseteq X$,

$$\mathrm{Coh}(X \text{ on } Z) \subseteq \mathrm{Coh}(X)$$

of coherent sheaves set theoretically supported on Z is a Serre subcategory.

(ii) $\mathrm{Coh}(Z) \subseteq \mathrm{Coh}(X)$ is not Serre.

(iii) R a coherent but not noetherian commutative ring.

$$\mathrm{Mod}_R^{\text{f.p.}} \subseteq \mathrm{Mod}_R$$

is not Serre. Ex: non-discrete valuation ring like $\mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p$.

Construction. $\mathcal{A} \subseteq \mathcal{B}$ Serre, can construct a new abelian category

$$\mathcal{B}/\mathcal{A} = \mathcal{B}[W^{-1}], \quad W = \{f : X \to Y : \ker(f), \operatorname{coker}(f) \in \mathcal{A}\}.$$

Ex. In (i), if $U = X \setminus Z$, then $\mathrm{Coh}(X)/\mathrm{Coh}(X \text{ on } Z) \simeq \mathrm{Coh}(U)$.

In fact, from Gabriel's thesis

$$\left\{ \begin{matrix} \text{specialization closed subsets} \\ \text{of } X \end{matrix} \right\} \xrightarrow[\mathrm{Coh}(X \text{ on } Z)]{\mathrm{supp}} \left\{ \begin{matrix} \text{Serre subcategories of} \\ \mathrm{Coh}(X) \end{matrix} \right\}$$

B. Some context, derived categories.

R a commutative ring.

$\mathrm{Perf}(R) \subseteq D(R)$.

<u>Def</u>. A <u>localization</u> of $D(R)$ is an adjoint pair with $\mathcal{C}$ Δed

$$\mathcal{D}(R) \underset{L}{\overset{U}{\leftrightarrows}} \mathcal{C}$$

where U is fully faithful. These are determined by

$$\begin{aligned} K = \ker(L) &\subseteq D(R) \\ \| \\ \{X \in D(R) &: LX \simeq 0\}. \end{aligned}$$

Such K are called <u>localizing</u> <u>subcats</u> of $D(R)$. They are f.f. Δed subcategories closed under arbitrary coproducts.

A localization is <u>smashing</u> if the following eq. conditions hold:

(a) L preserves compact objects ($\mathrm{Perf}(R)$),

(b) UL commutes w/ arbitrary coproducts,

(c) U commutes w/ arbitrary coproducts.

<u>Telescope conjecture</u>. If K is the kernel of a smashing localization, then K is generated by a set of compact objects of $D(R)$.

<u>Thm</u> (Neeman, Hopkins). R noetherian.

(a) $\left\{ \begin{matrix} \text{subsets of} \\ \operatorname{Spec} R \end{matrix} \right\} \;=\!=\; \left\{ \begin{matrix} \text{localizations of} \\ D(R) \end{matrix} \right\}$

(b) $\left\{ \begin{matrix} \text{specialization closed} \\ \text{subsets of} \\ \operatorname{Spec} R \end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix} \text{smashing loc. of} \\ D(R) \end{matrix} \right\}$; telescope is true.

Rems. (a) $Z \subseteq \operatorname{Spec} R$ is cut out by a f.g. ideal
$$I = (a_1, \dots, a_n).$$
The Koszul complex
$$\bigotimes_{i=1}^{n} \operatorname{cone}(R \xrightarrow{a_i} R)$$
is perfect and generates
$$D(\operatorname{Spec} R \text{ on } Z) \rightleftarrows D(\operatorname{Spec} R) \rightleftarrows D(U).$$

(b) Thomason proved that for any commutative R,
$$\left\{ \begin{matrix} \text{specialization closed} \\ \text{subsets of} \\ \operatorname{Spec} R \end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix} \text{thick subcats} \\ \text{of } \operatorname{Perf}(R) \end{matrix} \right\}.$$

(c) Telescope conjecture is typically false.

Let R be a perfect rank 1 valuation ring with maximal ideal $\mathfrak{m}$. Then, $k = R/\mathfrak{m}$ is also perfect. $\mathfrak{m}$ is torsion free, hence flat. Thus, $0 \to \mathfrak{m} \to R \to k \to 0$ is a flat resolution. Tensor with k:

$$\begin{tikzcd}
\mathfrak{m} \otimes_R k \arrow[r] \arrow[d, equal, "\substack{\text{Use} \\ \text{perfectness}}"'] & R \otimes_R k \arrow[r] \arrow[d, equal] & k \otimes_R k \arrow[r] \arrow[d, equal] & 0 \\
0 & k \arrow[r, equal] & k. &
\end{tikzcd}$$

Thus, $k \otimes^L_R k \simeq k$. Thus, $D(R) \underset{\otimes^L_R k}{\rightleftarrows} D(k)$ is a smashing localization. (Weird!) But, if $P \in \operatorname{Perf}(R)$ satisfies
$$P \otimes^L_R k \simeq 0,$$
then $P \simeq 0$. Hence, the kernel cannot be gen. by cpt. obj. of $D(R)$.

C. The almost categories.

The almost setup:

R commutative ring,

$I \subseteq R$ a flat ideal s.t.

$$I^2 = I.$$

(Includes $I = 0$ and $I = R$.)

<u>Lemma</u>. $I \otimes_R I \simeq I^2 = I$.

<u>Lemma</u>. $\mathrm{Mod}_{R/I} \subseteq \mathrm{Mod}_R$ is Serre.

<u>proof</u>. Just have to check closure under extensions. Let

$$0 \to M \to N \to P \to 0$$

be an ext in Mod_R with $M, P \in \mathrm{Mod}_{R/I}$. Fix $n \in N$ with image $\bar{n} \in P$. Let $a \in I$ and write $a = bc$ where $b, c \in I$. Then, $c\bar{n} = 0$, so $cn \in M$ and hence $b(cn) = 0$ whence $an = 0$.

<u>Def</u>. $\mathrm{Mod}_R^a := \mathrm{Mod}_R / \mathrm{Mod}_{R/I}$.

$$D(\mathrm{Mod}_R^a) = D(R)/D(R/I).$$

The six functors perspective.

$$\begin{tikzcd} \mathrm{Mod}_{R/I} \arrow[r, "i_*"] & \mathrm{Mod}_R \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \arrow[r, "j^*"] & \mathrm{Mod}_R^a \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"] \end{tikzcd}$$

— i^* left adj. to i_*.

— $i^!(M) = \mathrm{Hom}_R(R/I, M) = M[I]$ right adj. to i_*.

— $j_!$ fully faithful left. adj. to j^*.

— j_* fully faithful right adj. to j^*.

Rem. Kind of strange for $j_!$ to exist in the world of quasi-coherent sheaves.

Rem. Let U be the complement of $\mathrm{Spec}\, R/I \hookrightarrow \mathrm{Spec}\, R$.

Then,

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[rr] \arrow[dr] & & \mathrm{QCoh}(U) . \\ & \mathrm{Mod}_R^a \arrow[ur, dashed] & \end{tikzcd}$$

Bhargav suggests viewing Mod_R^a as q-c sheaves on a large (non-existent) open containing U.

The i adjoints exist for any ring map. The special thing here is that the image of ι_* is Serre.

<u>Def.</u> Set $\mathcal{A} \subseteq \mathrm{Mod}_R$ to be the full subcat of M s.t.

$$I \otimes_R M \xrightarrow{\simeq} M.$$

<u>Lemma.</u> $\mathcal{A} \simeq \mathrm{Mod}_R^a$. inclusion into Mod_R is $j_!$.

<u>proof.</u> j^*, j_* exist by formal nonsense. Momentarily, set

$$\mathrm{Mod}_R \xrightarrow{k^*} \mathcal{A}$$
$$M \longmapsto I \otimes_R M.$$

If $k^* M = 0$, then $M \cong R/I \otimes_R M$ and conversely.

Thus, $\ker(k^*) = \mathrm{Mod}_{R/I}$.

Let $k_* : \mathcal{A} \to \mathrm{Mod}_R$ by $M \longmapsto \mathrm{Hom}_R(I, M)$.

Then, $\mathrm{Hom}_R(M, \mathrm{Hom}_R(I,N)) \underset{\substack{\otimes\text{-Hom} \\ \text{adjunction}}}{\simeq} \mathrm{Hom}_R(I \otimes_R M, N) \simeq \mathrm{Hom}_{\mathcal{A}}(I \otimes_R M, N)$.

So, k_* is right adj. to k^*.

Given M in $\mathcal{A}$ we have a counit map

$$I \otimes_R \mathrm{Hom}_R(I, M) \longrightarrow M$$
$$a \otimes f \longmapsto f(a).$$

$$R\mathrm{Hom}(R/I, M) \longrightarrow M \longrightarrow R\mathrm{Hom}(I, M)$$

$$I \otimes^L_R R\mathrm{Hom}(R/I, M) \longrightarrow M \xrightarrow{\sim} I \otimes_R R\mathrm{Hom}(I, M).$$
$$\wr\!\!\mid$$
$$\underbrace{I \otimes^L_R R/I}_{\simeq 0} \otimes^L_{R/I} R\mathrm{Hom}(R/I, M) \simeq 0$$

$\mathrm{Hom}_{\mathcal{A}}(M, I \otimes_R N) \simeq \mathrm{Hom}_R(M, N)$.

How to show

$$R\mathrm{Hom}_R(M, R/I \otimes^L_R N) \simeq 0$$
$$\wr\!\!\mid$$
$$R\mathrm{Hom}_{R/I}(M \otimes^L_R R/I, R/I \otimes^L_R N)$$
$$\wr\!\!\mid$$
$$0$$

So, we have described $j_!$ and j_* <u>and</u> Mod_R^a.